# Math 249 Lecture 10 Notes

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## 1 Homogeneous and Power Sum Symmetric Functions

Recall from last lecture that we have 2 bases for the symmetric functions: the monomials $\{m_\lambda\}$ and the elementary symmetric functions $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_n}$, where $e_k = m_{1^k}$. We also have that

$$(z - x_1) \cdots (z - x_n) = z^n - e_1 z^{n-1} + e_2 z^{n-2} - \cdots \pm e_n.$$

**Example 1.1.** The function

$$\prod_{i<j} (x_i - x_j)^2 = \Delta(e_1, \dots, e_n)$$

is a symmetric function. This is the discriminant of a polynomial. So we can calculate discriminants using the $e_k$ terms.

### 1.1 Homogeneous symmetric functions

We introduce another basis for the symmetric functions: the homogeneous symmetric functions.

**Definition 1.1.** The *homogeneous symmetric functions* are the functions $h_\lambda$ for partitions $\lambda$ such that

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}, \qquad h_k = \sum_{|\lambda|=k} m_\lambda.$$

**Proposition 1.1.**

$$h_\lambda = \sum_\mu b_{\lambda,\mu} m_\mu,$$

*where $b_{\lambda,\mu}$ is the number of matrices with entries in $\mathbb{N}$, row-sums $\lambda$, and column-sums $\mu$.*

*Proof.* We use a similar matrix argument to the proof for expressing the $e_\lambda$ in terms of the $m_\mu$. Since symmetric functions must be invariant under permuting the variables, to find the coefficient $b_{\lambda,\mu}$, we need only find the coefficient in front of a single monomial in each $m_\mu$; we pick the monomial $x_1^{\mu_1} x_2^{\mu_2} \cdots x_\ell^{\mu_\ell}$. Each monomial in each $h_{\lambda_i}$ will be a product of variables with combined degree $\lambda_i$; then we can find the coefficient of a monomial in $h_\lambda$ by finding the number of ways to trace back where its $x_i$ terms could have come from in the product of the $h_{\lambda_i}$.

Now consider the following matrix (represented as a table):

| | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\cdots$ |
|---|---|---|---|---|
| $\lambda_1$ | 2 | 0 | 1 | $\cdots$ |
| $\lambda_2$ | 0 | 3 | 0 | $\cdots$ |
| $\lambda_3$ | 1 | 0 | 0 | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

We fill in this matrix as follows: Think of the column $j$ as picking an $x_j$ from some of $h_{\lambda_1}, h_{\lambda_2}, \ldots, h_{\lambda_\ell}$ (this is "where the $x_j$ term came from" when you multiply the $h_{\lambda_i}$), and place a 1 in the space $i, j$ if an $x_j$ term comes from $h_{\lambda_i}$ and a 0 otherwise. The product of all the $x_j$ terms in a column should give $x_j^{\mu_j}$ (since this is the monomial in $e_\lambda$ that we are looking at), so this is a matrix with column-sums $\mu_j$. Similarly, each monomial in $h_{\lambda_i}$ has variables with combined multiplicity $\lambda_i$, so the 1s in row $i$ should add to be $\lambda_i$. We have produced a bijection between these matrices and the number of ways to get $x_1^{\mu_1} x_2^{\mu_2} \cdots x_\ell^{\mu_\ell}$ from the product of the $h_{\lambda_i}$, so we are done. $\square$

### 1.1.1 Generating functions for the $e_k$ and $h_k$

Define the generating function for the $e_n$ terms: $E(t) = \sum_n e_n t^n$.

$$E(t) = \sum_n e_n t^n = \prod_i (1 + t x_i)$$

because $e_n$ is the sum of monomials where we choose $n$ different $x_i$ terms to multiply together. Also define the generating function for the $h_n$ terms: $H(t) = \sum_n h_n t^n$.

$$H(t) = \sum_n h_n t^n = \prod_i (1 + x_i t + x_i^2 t^2 + \cdots) = \prod_i \frac{1}{1 - x_i t},$$

where the second equality follows from the observation that multiplying the $(1 + x_i t + \cdots)$ terms keeps track of the number of powers of each $x_i$, where the powers of the $x_i$ terms correspond to a partition of $n$; then the coefficient of $t^n$ is the sum of all polynomials corresponding to partitions of $n$.

This gives us that

$$E(t) = 1/H(-t), \qquad E(t)H(-t) = 1,$$

and looking at the $t^k$ term in the latter series expansion gives us

$$e_k - e_{k-1}h_1 + e_{k-2} - \cdots + (-1)^k h_k = 0.$$

This gives us $e_k$ in terms of $h_1, \ldots, h_k$ and $h_k$ in terms of $e_1, \ldots, e_k$, and the expressions for each in terms of the other are the same! We also get that $\{h_\lambda : \lambda_1 \leq n\}$ is a basis for $\lambda(x_1, \ldots, x_n)$

Also, if we define for $\Lambda(x_1, x_2, \ldots)$ the map $\omega : \Lambda \to \Lambda$ sending $e_k \mapsto h_k$, we have $\omega_k^{-1} : \Lambda \to \Lambda$ given by $h_k \mapsto e_k$. This is because the expression for $h_k$ in terms of the $e_k$ is the exact same as the expression for $e_k$ in terms of the $h_k$. So $\omega^2 = \mathrm{id}_\Lambda$. This makes $\omega$ an involution.

## 1.2 Power sum symmetric functions

**Definition 1.2.** The *power sum symmetric functions* are the functions

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}, \text{ where } p_k = x_1^k + x_2^k + \cdots = m_{(k)}.$$

**Example 1.2.** We can calculate a few $p_\lambda$ in terms of our other bases.

$$p_2 = m_{(2)},$$

$$p_{(1,1)} = p_1^2 = m_{(2)} + 2m_{(1,1)},$$

$$e_2 = m_{(1,1)} = \frac{p_1^2 - p_2}{2},$$

$$h_2 = m_{(1,1)} + m_{(2,2)} = \frac{p_1^2 + p_2}{2}.$$

The power-sum symmetric functions are related to the homogeneous symmetric functions and the sizes of conjugacy classes of the symmetric group.

**Proposition 1.2.**

$$h_n = \sum_\lambda \frac{p_\lambda}{z_\lambda}, \text{ where } \frac{n!}{z_\lambda} = |C_\lambda|.$$

*Proof.* Use the generating function $H(t) = \sum_n h_n t^n = \prod_i \frac{1}{1-x_i t}$. Then

$$\log(H(t)) = \sum_i \log\left(\frac{1}{1-x_i t}\right) = \sum_i -\log(1-x_i t) = \sum_i \sum_{k=1}^\infty \frac{x_i^k t^k}{k} = \sum_{k=1}^\infty \frac{p_k t^k}{k}.$$

So, using the generating function for the sizes of conjugacy classes of the symmetric group (proved in lecture 3), we have

$$H(t) = \exp\left(\sum_k \frac{p_k t^k}{k}\right) = \sum_n \left(\sum_{|\lambda|=n} \frac{p_\lambda}{z_\lambda}\right) t^n. \qquad \square$$